CONTROLLABILITY OF STATIONARY LINEAR SYSTEMS WITH DELAY

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Abstract: This paper presents results on controllability of solution and control for linear stationary systems with delay. Analysis of controllability of linear stationary dynamical system with constant aftereffect in the case of commutative matrices of linear terms is conducted. Necessary and sufficient conditions for controllability are derived, general solution of the system is given and a control is build.

Keywords: Differential equation with delay, linear stationary system, commutative matrices, controllability.

1 INTRODUCTION

Particular results for functional-differential equations were obtained more than 250 years ago, and systematic development of the theory of such equations began only in the last 90 years. Before this time there were thousands of articles and several books devoted to the study and application of functional differential equations. However, all these studies are consider separate sections of the theory and its applications (the exception is well-known book L.E. Elsgolts, representing the full introduction to the theory, and its second edition published in 1971 in collaboration with S.B. Norkin [8]). There were no studies with single point of view on numerous problems in the theory of functional-differential equations until the book by J. Hale (1977) [5].

Interpretation of solutions of functional-differential equations as integral curve in the space RxC by N.N. Krasovskii (1959) [6] served as such single point. This interpretation is now widespread, proved useful in many parts of the theory, particularly the sections of the asymptotic behavior and periodicity of solutions. It clarified the functional structure of the functional-differential equations of delayed and neutral type, provided an opportunity to the deep analogy between the theory of such equations and the theory of ordinary differential equations and showed the reasons for not less deep differences of these theories. Classic work on the theory of functional, integral and integro-differential equations is a work by V. Volterra [10]. Biggest part of the results obtained during 150 years before works by V. Volterra were related to special properties of very narrow classes of equations.

In late 1930 and early 40s N. Minorsky (1942) very clearly pointed out the importance of considering the delay in feedback mechanism in his works on stabilizing the course of a ship and automatic control its movement. A.D. Myshkis [7] introduced general class of equations with retarded arguments and laid the foundation for general theory of linear systems. In 1972 Richard Bellman systematized ideas in [1] a broad applicability of equations that contain information about the past in such fields as economics and biology. He also presented a well-constructed theory of linear equations with constant coefficients and the beginning of stability theory. The most intensive development of these ideas presented in the book of R. Bellman and K. Cooke [2]. The book describes the theory of linear differential-difference equations with constant and variable coefficients.

The book by Pinney [9] is devoted to differential-difference equations, otherwise known as the equations with deviating argument. The focus of the book is on linear equations with constant coefficients, which are most often encountered in the theory of automatic control.

2 CONTROLLABILITY OF LINEAR STATIONARY SYSTEMS WITH ONE DELAY

Let X be the state space of a dynamical system; U be the set of the controlled effects (controls). Let $x = x(x_0, u, t)$ be the vector that characterizes the state of the dynamic system in the moment of time t by the initial condition x_0, $x_0 \in X$, $(x_0 = x|_{t=t_0})$ and by the control function u, $u \in U$.

Definition 2.1 *The state x_0 is called a controllable state in the class U (controlled state), if there are exist such control $u, u = u_{x_0} \in U$ and the number T, $t_0 \leq T = T_{x_0} < \infty$ that $x(x_0, u, T) = (0, ..., 0)^T$.*

Definition 2.2 *If every state $x_0, x_0 \in X$ of the dynamic system is controllable, then we say that the system is controllable (controlled system).*

Let us have the following Cauchy's problem:

$$\begin{aligned} &\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bu(t),\ t \in [0,T],\ T < \infty, \\ &x(0) = x_0,\ \ x(t) = \varphi(t),\ \ -\tau \leq t < 0, \end{aligned} \tag{1}$$

where $x = (x_1, ..., x_n)^T$ is the vector of phase coordinates, $x \in X$, $u(t) = (u_1(t), ..., u_r(t))^T$ is the control function, $u \in U$, U is the set of piecewise-continuous functions; A_0, A_1, B are constant matrices of dimensions $(n \times n)$, $(n \times n)$, $(n \times r)$ respectively, τ is the constant delay.
The state space Z of this system is the set of n-dimensional functions.

$$\{x(\theta),\ t-\tau \leq \theta \leq t\}. \tag{2}$$

The space of the n-dimensional vectors x (phase space X) is a subspace of Z. The initial state z_0 of the system (1) is determined by conditions

$$z_0 = \{x_0(\theta),\ x_0(\theta) = \varphi(\theta),\ -\tau \leq \theta < 0,\ x(0) = x_0\}. \tag{3}$$

The state $z = z(z_0, u, t)$ of the system (1) in the space Z in the moment of time t is defined by trajectory segment (2) of the phase space X.
Further consider that the movement system (1) goes $(t \geq 0)$ in the space of continuous function. Determining initial state of (3) of the function $\varphi(\theta)$ is piecewise-continuous.
In accordance with specified definitions of state (3) of the system (1) is controllable if there exist such control u, $u \in U$ that $x(t) \equiv 0$, $T - \tau \leq t \leq T$ when $T < \infty$.
The state (3) of the system (1) is relatively controllable if there exist a control u, $u \in U$, such that $x(T) = 0$ when $T < \infty$.

Definition 2.3 *The matrix function which has the form of a polynomial of degree k in intervals $(k-1)\tau \leq t \leq 0$,* **glued** *in knots $t = k\tau$,*

$$e_\tau^{At} = \begin{cases} \Theta, & -\infty < t < -\tau \\ I, & -\tau \leq t < 0 \\ \dots & \\ I + A\frac{t}{1!} + A^2\frac{(t-\tau)^2}{2!} + ... + A^k\frac{(t-(k-1)\tau)^k}{k!}, & (k-1)\tau \leq t < k\tau,\ k = 0, 1, 2, ... \end{cases}$$

where Θ is the zero matrix and I is the identity matrix, is called the delayed matrix exponential.

Delayed matrix exponentials (continuous or discrete) are used to solve boundary value problems and controllability problems cf. e.g. papers [3], [4].

3 REPRESENTATION OF SOLUTION OF THE CAUCHY PROBLEM

Let us have the system of differential equations with constant delay

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \quad x(t) \in R^n, \quad t \geq 0, \tag{4}$$

where $x(t) \equiv \varphi(t)$, if $-\tau \leq t \leq 0$ is the initial conditions, A_0, A_1 are square matrices with constant elements, $\varphi(t)$ is an arbitrary continuously differentiable initial vector-function.

Lemma 3.1 *Let matrices A_0 and A_1 be commutative, (i.e. $A_0 A_1 = A_1 A_0$). Then* $e_\tau^{A_0 t} A_1 = A_1 e_\tau^{A_0 t}$, $t \geq 0$.

Using all the above statements we obtain the explicit form of the fundamental matrix of the system (4) for commutative matrices A_0, A_1.

Theorem 3.2 *Let matrices A_0, A_1 of system (4) be commutative and let there exist A_0^{-1}. Then the matrix*

$$\mathrm{X}_0(t) = C e^{A_0 t} e_\tau^{D(t-\tau)},$$

where $D = e^{-A_0 \tau} A_1, C = (I + A_1 A_0^{-1}), t \geq 0$ is the solution of the system (4) satisfying the initial conditions $\mathrm{X}_0(t) \equiv I, \quad -\tau \leq t \leq 0.$

Theorem 3.3 *Let matrices A_0, A_1 of system (4) be commutative. Then the solution of the Cauchy problem for system (4) with initial conditions $x(t) \equiv \varphi(t), -\tau \leq t \leq 0$ has the form*

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds$$

Let us have a linear heterogeneous system with delay

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + f(t), \tag{5}$$

where $f(t)$ is some vector-function, $f(t) = (f_1(t), .., f_n(t))^T$.

Theorem 3.4 *Let matrices A_0, A_1 of system (5) be commutative. Then the solution $\overline{x(t)}$ of the heterogeneous system (5) which satisfies zero initial conditions has the form*

$$\overline{x(t)} = \int_{0}^{t} e^{A_0(t-\tau-s)} e_\tau^{D(t-2\tau-s)} f(s)ds, \quad t \geq 0.$$

Theorem 3.5 *. The solution of heterogeneous system (5) which satisfies the initial conditions $x(t) \equiv \varphi(t), -\tau \leq t \leq 0$ has the form*

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int_{0}^{t} \mathrm{X}_0(t-\tau-s)f(s)ds. \tag{6}$$

4 THE CONTROL CONSTRUCTION FOR SYSTEM WITH COMMUTATIVE MATRICES

Let us have the control system of differential equations

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bu(t), \tag{7}$$

where $x(t) \in R^n, t \geq 0, \tau > 0$, $x(t) \equiv \varphi(t)$ if $-\tau \leq t \leq 0$, A_0, A_1 are square matrices, B is the constant matrix of dimension $(n \times n)$, $u(t) = (u_1(t), .., u_n(t))^T$ is the control vector-function.

Theorem 4.1 *For a linear stationary system with delay (7) to be controllable it is necessary that the following condition holds:* $t \geq (k-1)\tau, k = 1,2,3,\ldots$ *and* rank $S_k = n$, *where*

$$S_k = \{B; e^{-A_0\tau}A_1B; e^{-2A_0\tau}A_1^2B; ...; e^{-(k-1)A_0\tau}A_1^{k-1}B\}.$$

Theorem 4.2 *System (7) is controllable for* $t \geq (k-1)\tau, k = 1,2,3,\ldots,$ *if holds:* rank $Q_k = n$, *where* $Q_k = \{B; [A_0 e^{A_0\tau} e_\tau^{D\tau} + A_1]B; ...; [A_0^{k-1} e^{A_0(k-1)\tau} e_\tau^{D(k-1)\tau} + A_1^{k-1}]B\}$, *where* $D = e^{-A_0\tau}A_1$.

Theorem 4.3 *Let* $t_1 \geq (k-1)\tau, k = 1,2,3,\ldots$ *and the necessary and sufficient conditions for controllability be implemented:* rank $S_k =$ rank$\{B; e^{-A_0\tau}A_1B; e^{-2A_0\tau}A_1^2B; ...; e^{-(k-1)A_0\tau}A_1^{k-1}B\} = n$,
rank $Q_k =$ rank $\{B; [A_0 e^{A_0\tau} e_\tau^{D\tau} + A_1]B; \ldots; [A_0^{k-1} e^{A_0(k-1)\tau} e_\tau^{D(k-1)\tau} + A_1^{k-1}]B\} = n$, *where* $D = e^{-A_0\tau}A_1$.
Then the control function can be taken as

$$u(s) = [\mathrm{X}_0(t_1 - \tau - s)B]^T \left[\int_0^{t_1} \mathrm{X}_0(t_1 - \tau - s)BB^T[\mathrm{X}_0(t_1 - \tau - s)]^T ds \right]^{-1} \mu,$$

where $\mu = x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int\limits_{-\tau}^{0} \mathrm{X}_0(t_1 - \tau - s)\varphi'(s)ds.$

5 EXAMPLE

Example 1. Consider a system of two differential equations with a constant delay:

$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + Bu(t)$, where $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

As we see $\tau = 1, n = 2$ and $A_0A_1 = A_1A_0$. We want to know whether this system is controllable in the moment of time $t_1 = 3$. Let us check the necessary condition. First, we find the matrix $S_3 = \left\{B, e^{-A_0\tau}A_1B, e^{-2A_0\tau}A_1^2B\right\} = \left\{B, (e^{-1}I)A_1B, (e^{-2}I)A_1^2B\right\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e^{-2} & 0 \\ 0 & 0 \end{pmatrix} \right\}$

We have $rank(S_3) < 2$ so the system is not controllable for $t_1 = 3$.

Example 2. Let us have the differential equation of 2^{nd} degree with a constant delay:

$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + Bu(t)$, where $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

As we see $\tau = 1, n = 2$ and $A_0A_1 = A_1A_0$. We want to know whether this system is controllable in the moment of time $t_1 = 2$. Let us check the necessary and sufficient conditions:

$rank(S_2) = rank(\left\{B, e^{-A_0\tau}A_1B\right\}) = 2, \quad rank(Q_2) = rank(\left\{B, [A_0 e^{A_0\tau} e_\tau^{D\tau} + A_1]B\right\}) = 2.$

It is easy to see that the necessary and sufficient conditions for controllability hold ($rank(B) = 2$), so the system is controllable in time moment $t_1 = 2$.

Let us use the solution of the Cauchy problem in general case (6). If $u(t) = 0$, we have that for $1 \le t \le 2, x(t) = 0$, so $x(2) = 0$. Let us construct such control function, that system in time moment $t_1 = 2$ be in point $x_1 = (1,1)^T$, using initial condition $x_0(t) \equiv \varphi(t) = (0,0)^T, -\tau \le t \le 0$. By result of the theorem (4.3) we write:

$$u(t) = [e^{A_0(t_1-\tau-t)} e_\tau^{D(t_1-2\tau-t)} B]^T [\int_0^{t_1} e^{A_0(t_1-\tau-s)} e_\tau^{D(t_1-2\tau-s)} BB^T [e^{A_0(t_1-\tau-t)} e_\tau^{D(t_1-2\tau-t}]^T ds]^{-1} \mu,$$

$\mu = x_1 - e^{A_0(t_1)} e_\tau^{D(t_1)} \varphi(-\tau) - \int_{-\tau}^{0} e^{A_0(t_1-\tau-s)} e_\tau^{D(t_1-2\tau-s)} \varphi'(s) ds$. So, we have

$$u(t) = [e^{(1-t)} e_1^{D(-t)}]^T [\int_0^2 e^{(1-s)} e_1^{D(-s)} [e^{(1-s)} e_1^{D(-s)}]^T ds]^{-1} \mu, \mu = (1,1)^T.$$

Finally, we get $u(t) = e^{-t}(0.7e^3 - 0.7e^2 t + 0.35et^2, \quad -0.1e^3 - 1.1e^2 t + 2.35et^2)^T$. And for this control function we have $x(3) = (1.017, 0.986)^T$.

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