APPLICATION OF WAŻEWSKI'S TOPOLOGICAL METHOD

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ABSTRACT

This paper is devoted to Ważewski's topological method in the form known for ordinary differential equations. This method is applied to certain classes of differential equations which solutions remain in a given set on their right-hand maximal interval of existence.

1 INTRODUCTION AND PRELIMINARIES

Topological methods are frequently used in proving results on qualitative properties of differential equations, especially in problems of the existence of solutions satisfying some boundary value data. They are usually based on fixed point theorems or on properties of the Brouwer and Leray-Schauder degrees (see[1,2]).

In 1947, Tadeusz Ważewski[3] published the paper in which he presented a new topological method for proving the existence of solutions remaining in a given set. In this paper we show some examples of applications of this method for ordinary differential equations. Now we give a short summary of Ważewski's topological method.

Let $f(t,y)$ be a continuous function defined on an open (t,y)-set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$, Ω^0 be an open set of Ω, $\partial\Omega^0$ the boundary of Ω^0 with respect to Ω and $\overline{\Omega}^0$ the closure of Ω^0 with respect to Ω. Consider the system of ordinary differential equations

$$y' = f(t,y) \tag{1}$$

and suppose that system (1) satisfies conditions of the existence and uniqueness of solutions in Ω.

Definition 1. *The point $(t_0,y_0) \in \Omega \cap \partial\Omega^0$ is called an egress (or an ingress point) of Ω^0 with respect to system* (1) *if, for every fixed solution of system* (1)*, $y(t_0) = y_0$, there exists an $\varepsilon > 0$ such that $(t,y(t)) \in \Omega^0$ for $t_0 - \varepsilon \le t < t_0$ ($t_0 < t \le t_0 + \varepsilon$). An egress point (ingress point) (t_0,y_0) of Ω^0 is called a strict egress point (strict ingress point) of Ω^0 if $(t,y(t)) \notin \overline{\Omega}^0$ on interval $t_0 < t \le t_0 + \varepsilon_1$ ($t_0 - \varepsilon_1 \le t < t_0$) for an ε_1.*

Definition 2. *An open subset Ω^0 of the set Ω is called an (u,v)-subset of Ω with respect to system* (1) *if the following conditions are satisfied:*

(1) There exist functions $u_i(t,y) \in C^1(\Omega,\mathbb{R})$, $i=1,\ldots,m$ and $v_j(t,y) \in C[\Omega,\mathbb{R}]$ $j=1,\ldots,n$, $m+n>0$ such that

$$\Omega_0 = \{(t,y) \in \Omega : u_i(t,y) < 0,\ v_j(t,y) < 0\ \forall i,j\}.$$

(2) $\dot{u}_\alpha(t,y) < 0$ holds for the derivatives of the functions $u_\alpha(t,y)$, $\alpha = 1,\ldots,m$ along trajectories of system (1) *on the set*

$$U_\alpha = \{(t,y) \in \Omega : u_\alpha(t,y) = 0,\ u_i(t,y) \le 0,\ v_j(t,y) \le 0,\ \forall j \text{ and } i \ne \alpha\}.$$

(3) $\dot{v}_\beta(t,y) > 0$ holds for the derivatives of the functions $v_\beta(t,y)$, $\beta = 1,\ldots,n$ along trajectories of system (1) *on the set*

$$V_\beta = \{(t,y) \in \Omega : v_\beta(t,y) = 0,\ u_i(t,y) \le 0,\ v_j(t,y) \le 0,\ \forall i \text{ and } j \ne \beta\}.$$

The set of all points of egress (strict egress) is denoted by Ω_e^0 (Ω_{se}^0).

Lemma 1. *Let the set Ω_0 be a (u,v)-subset of the set Ω with respect to system* (1)*. Then*

$$\Omega_{se}^0 = \Omega_e^0 = \bigcup_{\alpha=1}^{m} U_\alpha \setminus \bigcup_{\beta=1}^{n} V_\beta.$$

Definition 3. *Let X be a topological space and $B \subset X$.*

Let $A \subset B$. A function $r \in C(B,A)$ such that $r(a) = a$ for all $a \in A$ is a retraction from B to A in X.

The set $A \subset B$ is a retract of B in X if there exists a retraction from B to A in X.

Theorem 1. (Ważewski's theorem . *Let Ω^0 be some (u,v)-subset of Ω with respect to system* (1)*. Let S be a nonempty compact subset of $\Omega^0 \cup \Omega_e^0$ such that the set $S \cap \Omega_e^0$ is not a retract of S but is a retract of Ω_e^0. Then there is at least one point $(t_0,y_0) \in S \cap \Omega_0$ such that the graph of a solution $y(t)$ of the Cauchy problem $y(t_0) = y_0$ for* (1) *lies in Ω_0 on its right-hand maximal interval of existence.*

2 APPLICATIONS

Consider the system of ordinary differential equations

$$x' = f(t,x,y), \quad y' = g(t,x,y), \tag{2}$$

where $f,g \in C(\Omega,\mathbb{R})$, $\Omega \subset \mathbb{R}^3$ is an open set and suppose that system (2) satisfies conditions of the existence and uniqueness of solutions in Ω .

Theorem 2. *Assume that f and g satisfy for any t*

$$xf(t,x,y) > 0 \quad \text{if } |x| = 1 \text{ and } |y| \le 1 \tag{3}$$

and

$$yg(t,x,y) < 0 \quad \text{if } \ |x| \le 1 \text{ and } |y| = 1. \tag{4}$$

Then there exists $x_0 \in (-1,1)$ *such that the solution of (2) with* $x(0) = x_0$ *and* $y(0) = 0$ *satisfies* $|x(t)| < 1$ *and* $|y(t)| < 1$ *for all* $t \ge 0$*.*

Proof. Define the set

$$\Omega_0 = \{(t,x,y) \in \mathbb{R}^3 : |x| < 1, \ |y| < 1\}.$$

The conditions (3),(4) tell us the set

$$U = \{(t,x,y) \in \mathbb{R}^3 : |x| \le 1, \ |y| = 1\}$$

is the set of strict ingress points of Ω_0 and, similarly, the set

$$V = \{(t,x,y) \in \mathbb{R}^3 : |x| = 1, \ |y| \le 1\}$$

is the set of strict egress points of Ω_0.

Take

$$Z = \{(0,x,0) \in \mathbb{R}^3 : |x| \le 1\}.$$

Then

$$Z \cap V = \{(0,x,0) \in \mathbb{R}^3 : |x| = 1\}.$$

Since $Z \cap V$ consists of two distinct points and Z is a connected set, $Z \cap V$ is not a retract of Z. On the other hand, $Z \cap V$ is a retract of V since the points from the set $\{(0,-1,y) : |y| \le 1\}$ can be continuously deformed to the point $(0,-1,0)$ and the points from the set $\{(0,1,y) : |y| \le 1\}$ can be continuously deformed to the point $(0,1,0)$. Hence by Ważewski's theorem we get the assertion of Theorem 1.

Consider the following simple Cauchy initial problem

$$x' = x, \quad x(1) = 0. \tag{5}$$

Take

$$\Omega_0 = \{(t,x) \in \mathbb{R}^2 : u(t,x) = x^2 - 1 < 0, \ v(t,x) = 1 - t < 0\}$$

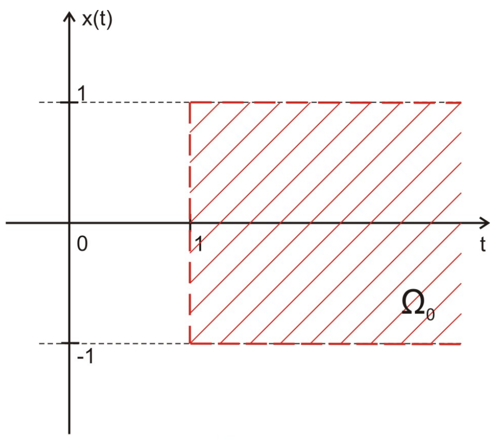

Figure 1.

Put $\quad U(t,x) = \{(t,x) \in \mathbb{R}^2 : |x| = 1,\ 1-t \leq 0\}$ (see Figure 2.).

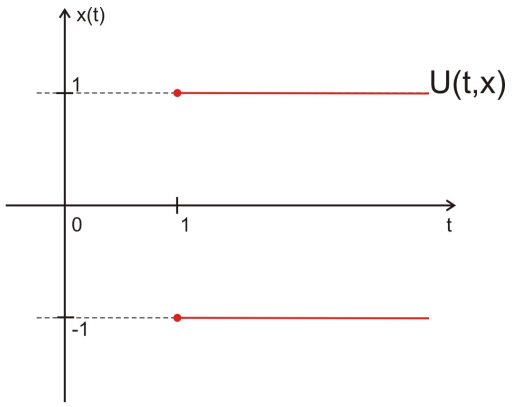

Figure 2.

$V(t,x) = \{(t,x) \in \mathbb{R}^2 : |x| \leq 1,\ t = 1\}$ (see Figure 3.).

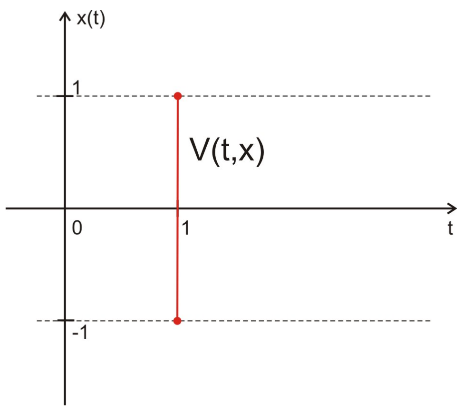

Figure 3.

Then we get

$$\dot{u}(t,x) = (x^2-1)' = 2xx' = 2x^2 > 0$$

Then all points on $U(t,x)$ are the strict egress points (see Figure 4.).

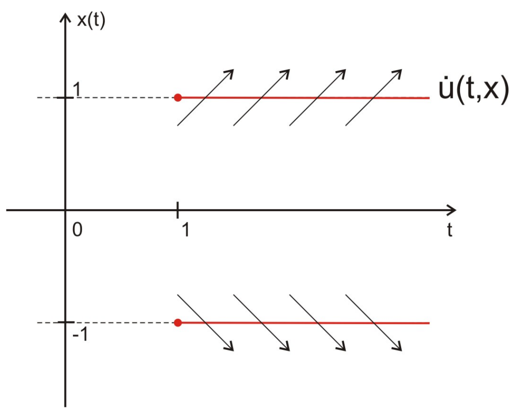

Figure 4.

Similarly,

$$\dot{v}(t,x) = (1-t)' = -1 < 0$$

Hence all points on $V(t,x)$ are the strict ingress points (see Figure 5.).

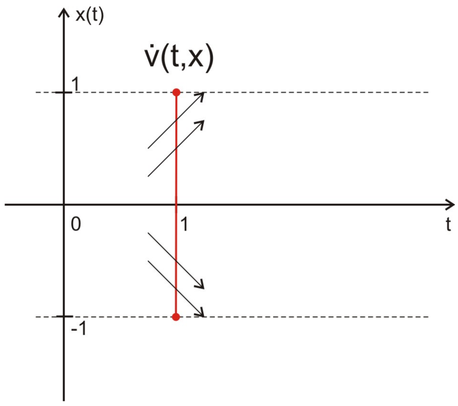

Figure 5.

It is obvious that the general solution of (5) has the form $x(t) = Ce^t$ and the particular solution $x(t) = 0$. From Figure 6. and according to Wazewski's theorem there exists a solution of (5) lying in Ω_0 on its right-hand maximal interval of existence that is $x(t) = 0$, which validates our computations.

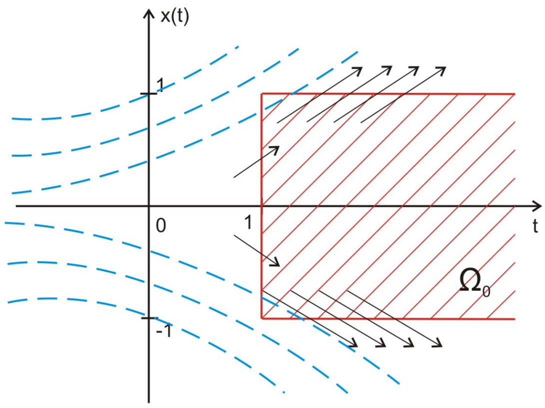

Figure 6.

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