PARTIAL DIFFERENTIAL EQUATIONS IN IMAGE PROCESSING

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ABSTRACT

The paper deals with less known methods of image processing, i.e. methods that employ partial differential equations. The methods described in the overview article are methods for image restoration (energy method, total variation restoration, heat equation, nonlinear diffusion, and Osher-Rudin shock filter for the enhancement of edges in an image) and methods for image segmentation (application of the level set method as a segmentation approach to complex shapes). The principles of several methods, their mathematical models, and examples of image processing are given.

1. INTRODUCTION

Improving the quality of images has recently come to be one of very important technical problems, which concerns not only technical fields but also many fields of science and, in particular, the field of medical diagnostic methods. The obtained image is degraded by deterministic and random phenomena. The former may be due to, for example, a wrong adjustment of the optic mechanism or some motion. Random phenomena are taken to mean noise in the image or errors in signal transmission. Random noise is characterized by probability distribution.

Many methods have been proposed for image processing; the less known but efficient methods include methods that employ partial differential equations (PDE). These methods can be used for restoration, segmentation, and exacting applications such as *inpainting* (painting in the damaged part of image), decomposition of image into geometric shapes and textures, sequence analysis, and image classification. This overview article is concerned with the application of methods of restoration and segmentation.

For image restoration, the energy method and the total-variance minimization are described. The latter does not smooth edges as the energy methods does during noise filtering. Other methods consist in a direct application of PDE as equations of heat conduction and non-linear diffusion. These methods serve to filter noise, in other words they filter the higher frequency components. Yet another type of image restoration method is PDEs for the enhancement of edges in the image, that is to say high-pass filters. The Osher-Rudin shock filter is give.

For image segmentation, the level set method is described. Here, the principle is the evolution of contours in the image, which draw close to object edges with time. The contour is the result of a 3D function intersecting with the zero level. An advantage of the level set method in comparison with active contours is the change in the curve shape based on the level set function. The method can thus be applied to very complex shapes in the image.

2. MATHEMATICAL MODELS

2.1. IMAGE RESTORATION

Image restoration essentially consists in removing noise or reducing the above described phenomena. Let us take a general model:

$$u_0 = Ru + \eta\,, \tag{1}$$

for an image obtained, where η is Gaussian white additive noise, and R is a linear operator representing a blur (most frequently a convolution). The aim of the methods is to reconstruct the original image describing a real scene. This an inversion problem and we can only obtain an approximation of original image u , which we examine over region Ω.

The Energy Method

The classical solution to this inversion problem consists in adding a regularization member to the energy difference between the original and the restored image. We solve generally the following minimization problem:

$$E(u) = \frac{1}{2}\int_{\Omega} |u_0 - Ru|^2 \,\mathrm{d}x + \lambda \int_{\Omega} \phi(|\nabla u|)\mathrm{d}x\,. \tag{2}$$

The first member is a measure of agreement between the obtained and the original image, and the second element is the smoothing element. We are seeking an image whose total gradient is small – with the noise removed. The parameter λ is a positive constant. The function Φ must be chosen such that the resultant image is formed by regions with a constant brightness value separated by sharp edges. Energy minimization (2) corresponds to the Euler-Lagrange equation:

$$R^* Ru - \lambda \,\mathrm{div}\left(\frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u \right) = R^* u_0\,, \tag{3}$$

where R^* is a matrix adjunct to R. For simplification, we choose R = **I**.

Total variation restoration

An approach similar to that of the energy methods is the minimization of total variation. There is no member with energy here, only the smoothing element, which exactly is formed by total variation defined as:

$$TV(u) = \int_{\Omega} |\nabla u(x,y)| \,\mathrm{d}x\mathrm{d}y\,. \tag{4}$$

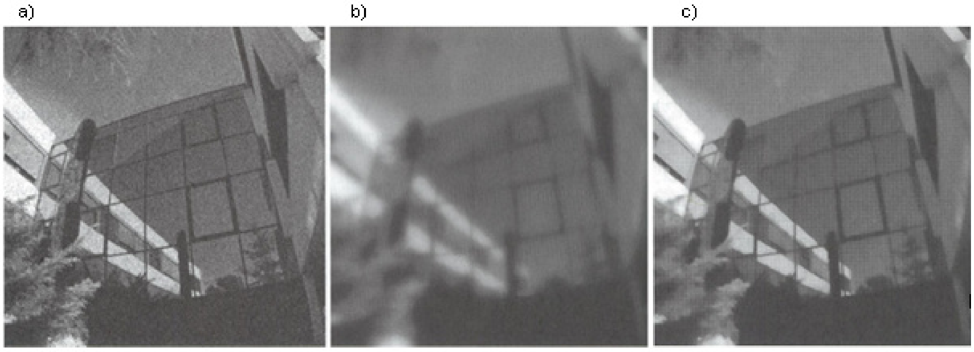

Fig. 1: Processing by energy method, a) original image, b) L^2 gradient normal as the smoothing member, c) L^1 gradient normal as the smoothing member

By minimizing (4) we will obtain a partial differential equation in the form:

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right). \tag{5}$$

This method is already capable of preserving edges in the image better than the energy method with a simple smoothing member (L^2 gradient normal).

The Heat Equation, Nonlinear Diffusion

The method of heat conduction equation and non-linear diffusion applies differential equations to an image signal. The heat conduction equation can be written in the form:

$$\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = 0\,, \qquad t \geq 0\,, x \in R^2 \tag{6}$$

And the equation of non-linear diffusion in the form:

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(c\left(|\nabla u|^2\right)\nabla u\right). \tag{7}$$

Equation (7) corresponds to signal processing by a Gaussian filter with core:

$$G_\sigma(x) = \frac{1}{2\pi\sigma^2} \cdot \exp\left(-\frac{|x|^2}{2\sigma^2} \right), \tag{8}$$

with a relation holding between the number of iterations and the standard filter deviation $\sigma = \sqrt{2t}$. The main disadvantage of this method is that it smoothes the image inclusive of sharp edges. This drawback is solved by the equation of non-linear diffusion if the function c has been selected correctly (for $c \equiv 1$ the diffusion equation changes to the heat conduction equation). In order that filtering should not smooth edges, the function must acquire values close to zero for high gradient values (edge) and values close to one for low gradient values (noise), which will ensure that smoothing near the edges will stop. For example:

$$c(s) = \frac{1}{\sqrt{1+s}}\,. \tag{9}$$

The Osher-Rudin Shock Filter

The Osher-Rudin filter belongs to the category of high-pass filters. It thus serves to enhance edges in the image, to focus. The equation of the filter is of the following form:

$$\frac{\partial u}{\partial t} = -|\nabla u| F(L(u)), \tag{10}$$

Where F is a function that fulfils the conditions:

$$F(0) = 0,\ sign(s)F(s) > 0, \quad s \neq 0, \tag{11}$$

and L is the edge detector (zero-cross). An example can be seen in the Laplacian (sum of second derivatives with respect to co-ordinates). The result of filtering by this filter is shown in Fig. 2 (taken over from [1]). The filter creates from blurred regions homogeneous regions with sharp transitions.

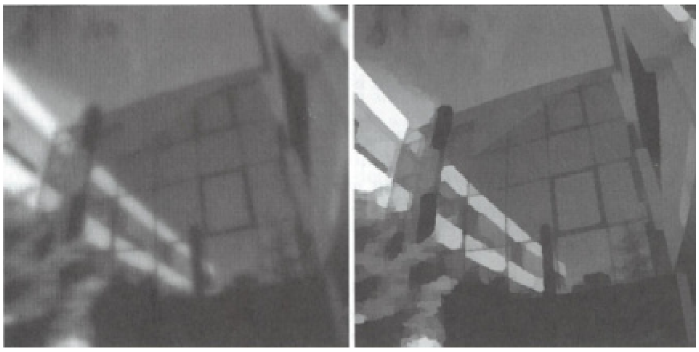

Fig. 2: Enhancement of edges in an image, using the Osher-Rudin shock filter; left – original image, right – focused image

2.2. IMAGE SEGMENTATION

Compared with classical methods such as segmentation via thresholding or via following the object boundaries, the level set method is a completely different approach. By its principle, it resembles the method of active contours, but here we change the shape of curves by means of the level set function. The method is therefore appropriate for more complex shapes of objects in the image.

Level-Set Method

The principle of this method consists in shaping a function that is one dimension larger than the segmented image. The contour is thus the result of such a function intersecting the plane at zero level. Fig. 3 (taken over from [2]) shows its application in processing a magnetic resonance image, specifically the segmentation of white cerebral cortex. The equation of the evolution of level-set function is defined as follows:

$$\frac{\partial \phi}{\partial t} + F|\nabla \phi| = 0\,, \tag{12}$$

where F is the so/called velocity function, which is made up of three members:

$$F = F_0 + F_{\mathrm{curv}} + F_{\mathrm{ext}}\,, \tag{13}$$

The first member, F_0, gives the constant velocity of zero contour in the direction of the normal, F_{curv} is a velocity dependent on local curvature, and F_{ext} is the velocity in the outer vector field **V**. Substituting equation (13) into (12) will yield an extended equation for the evolution of level set function:

$$\frac{\partial \phi}{\partial t} + \beta\left(\vec{V} \cdot \nabla \phi\right) + F_0 \left|\nabla \phi\right| = \varepsilon\kappa\left|\nabla \phi\right| . \tag{14}$$

Finally, it is necessary to choose suitably the function F_0 and the vector field **V**. We choose the function F_0 as an edge detector while the field vector is chosen such that the contour at the place of edge stops and does not continue expanding in the direction of normal vector.

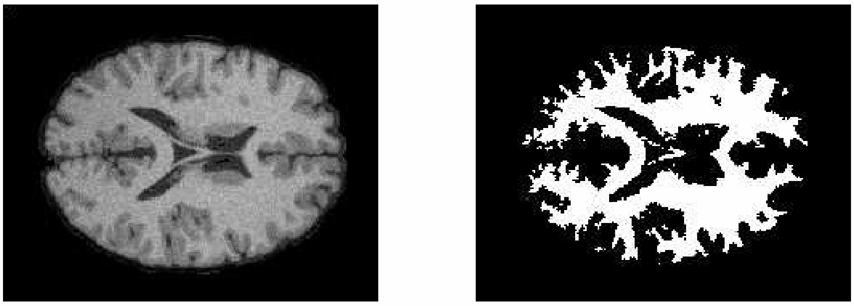

Fig. 3: Segments of an MR image: white cerebral cortex

As can be seen from Fig. 3, contours can be used to segment in the image objects of greater complexity by means of level set.

3. CONCLUSION

The methods described offer a less known approach to the region of image processing when compared with currently used methods of filtering (filtering in the spectral region or filtering the wavelet transform coefficients) and segmentation (thresholding, following the boundaries, etc.). The methods are more time-consuming but also much more variable. A typical example is the possibility of choosing the smoothing member of the energy method or the choice of the non-linear diffusion member controlling the efficiency of noise elimination on the basis of the gradient in the image. Individual equations can be re-written arbitrarily, depending on the properties of the image, noise, and possibly also other sources of image degradation. Described methods will be used to the NMR image processing in my thesis in future.

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